Elites in Social Networks: An Axiomatic Approach*

Chen Avin, Zvi Lotker, David Peleg, Yvonne-Anne Pignolet, Itzik Turkel

Abstract Recent evidence shows that in many societies the relative sizes of the economic and social *elites* are continuously shrinking. Is this a *natural* social phenomenon? We try to address this question by studying a special case of a core-periphery structure composed of a social *elite*, namely, a relatively small but well-connected and highly influential group of powerful individuals, and the rest of society, the *periphery*. Herein, we present a novel axiom-based model for the mutual influence between the elite and the periphery. Assuming a simple set of axioms, capturing the elite's *dominance*, *robustness* and *compactness*, we are able to draw strong conclusions about the elite-periphery structure. In particular, we show that the elite size is *sublinear* in the network size in social networks adhering to the axioms. We note that this is in controversy to the common belief that the elite size converges to a linear fraction of society (most recently claimed to be 1%).

1 Introduction

In his book *Mind and Society* [22], Vilfredo Pareto wrote what is by now widely accepted by sociologists: "Every people is governed by an *elite*, by a chosen element of the population". Indeed, with the exception of some rare examples of utopian or totally egalitarian societies, almost all societies exhibit an (often radically) uneven distribution of power, influence, and wealth among their members, and, in particular, between the *elite* and its complement, sometimes referred to as the *masses*. Typically, the elite is small, powerful and influential, whereas the complementary part of society is larger, less organized, and less dominant.

Looking more closely at social networks, the distinction between the elite and the rest of society can be viewed as a special case of a more general division that occurs in most complex networks, usually referred to as a *core-periphery* partition of the network [5]. The core-periphery structure is ar-

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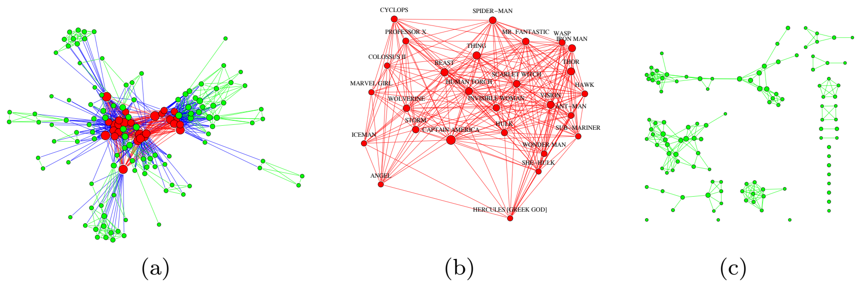

Fig. 1 Fictional illustrative example: the social network of the Marvel's superheroes. Two heroes are linked if they appeared together in many comic book titles [1]. (a) The network (139 superheroes, 924 edges), partitioned into an elite (red vertices and internal edges) and a periphery (green vertices and internal edges). Blue "crossing" edges connect elite and periphery vertices. (b) The (dense) elite subgraph (27 vertices, 252 edges). (c) The (sparser) periphery subgraph (112 vertices, 249 edges).

guably the most high-level structure of society, and the problem of identifying this partition and understanding its basic properties has recently received increasing attention [19, 24, 26]. Generally speaking, the core vertices are more highly connected and feature higher centrality values than the periphery vertices; these properties are naturally shared by elites in social networks.

However, we argue that elites have some additional significant properties, which distinguish them as a special class of cores worthy of independent study. In particular, these properties imply two notable characteristics of elites, namely, that they are relatively *small* and that they possess a *disproportionate* fraction of the power, resources, and influence in society.

In this paper we study the properties of social elites. Our main contribution is a characterization of elites, i.e., a set of properties (formulated as "axioms") concerning influence and density that any elite must possess. We stress that we do not claim these axioms hold for every core-periphery partition, nor do we claim that every social network admits a core-periphery partition that satisfies the axioms; in fact, it is easy to find examples of both real-life complex networks and classical evolutionary network models in which our axioms are not met by any core-periphery partition. Rather, we focus on the class of social networks that *do* admit core-periphery partitions that satisfy our axioms, referred to hereafter as the class of *elite-centered* social networks.

A small illustrative example of the terms we use is provided in Figure 1. It presents the network of the top 139 Marvel [1] superheroes and the 924 links interconnecting them, partitioned into an elite and a periphery as shown by different vertex colors. Two striking features can be clearly observed in this figure. First, the elite (containing, e.g., Captain America, Spiderman, and Thor), depicted in 1(b), is dense and organized, while the periphery, presented in 1(c), is much sparser and less structured. Second, the size of the core is "only" 27 vertices, with 112 vertices in the periphery. Note that despite this considerable size difference, the elite and the periphery have almost the same number of internal edges (≈ 250).

Our axiomatic characterization does not lead to pinpointing a single definition for the elite in a given social network. However, it is powerful enough to allow us to derive several conclusions concerning basic properties of the elite in society. Our main conclusion applies to the *size* of the elite. Recent reports show that the gap between the richest people and the masses keeps increasing, and that decreasingly fewer people amass more and more wealth [14, 21]. The question raised by us is: can society help it, or is this phenomenon an unavoidable by-product of some inherent natural properties of society? We claim that in fact, one can predict the shrinkage of elite size over time (as a fraction of the entire society size) based on the very nature of social elites. In particular, in our model, such shrinkage is the natural result of a combination of two facts: First, *society grows*, and second, *elites are denser* than peripheries (informally, they are much better connected). Combining these facts implies that the fraction of the total population size comprising dense elites will decrease as the population grows with time. We prove this formally in Theorem 2.

A dual question we are interested in concerns the stable size of the elite in a growing society: How small can the elite be while still maintaining its inherent properties? In general, elites of constant size can exist in societies where influence might be sharply asymmetric. In contrast, we show that if the social network is unweighted and undirected then an elite cannot be smaller than $\Omega(\sqrt{m})$, where m is the number of network edges.

Supporting empirical results are presented in [4].

2 An Axiomatic Approach

The common approach to explaining empirical results on social networks is based on providing a new concrete (usually random) *evolutionary model* and comparing its predictions to the observed data. In contrast, we follow an *axiomatic approach* to the questions at hand. This approach is based on postulating a small set of axioms, capturing certain expectations about the network structure and the basic properties that an elite must exhibit in order to maintain its power in the society. Our axioms are inspired by elite definitions like the one from Wikipedia, by which:

> *"In political and sociological theory, an elite is a small group of people who control a disproportionate amount of wealth or political power".*

To conceptualize these informal definitions, we employ the fundamental notion of *influence* among groups of vertices, and propose three independent properties related to the influence between the elite and the periphery. The underlying assumption is that the excess influence of the elite allows it on the one hand to control the rest of the population, and on the other to protect its members from being controlled by others outside the elite. We refer to these two properties as *dominance* and *robustness* respectively. In addition, the "wealth" is shared by the elite few, implying that on average, the elite members hold much more influence than individuals in the periphery. We refer to this property as *compactness*. We characterize *elite-centered* social networks as the class of social networks that admit core-periphery partitions satisfying these three properties. Next we make these properties more formal.

Influence and Core-Periphery Partition. We consider *influence* to be a measurable quantity between any two *groups* of people from the population, X and Y, denoted by $\mathcal{I}(X,Y)$. The groups X and Y are not necessarily distinct, and we are also interested in the *internal influence* exerted by the vertices of a group X on themselves, referred to by $\mathcal{I}(X,X)$. We provide the abstract notion of influence in social networks a concrete interpretation based on edge weights. We also assume that every individual has a self-opinion (modeled as a *self-loop*.) We denote the set of core vertices by $\mathcal{C}$ and the rest of society (the periphery) by $\mathcal{P}$. We call the pair $(\mathcal{C},\mathcal{P})$, which satisfies $\mathcal{C}\cap\mathcal{P}=\emptyset$ and $\mathcal{C}\cup\mathcal{P}=V$, a *core-periphery* partition, and study the four influence quantities $\mathcal{I}(\mathcal{C},\mathcal{C})$, $\mathcal{I}(\mathcal{P},\mathcal{P})$, $\mathcal{I}(\mathcal{C},\mathcal{P})$ and $\mathcal{I}(\mathcal{P},\mathcal{C})$.

Formally, we model a social network as a directed, weighted graph $G=(V,E,\omega)$, with a set V of n vertices representing the members of society, connected by a set $E\subseteq V\times V$ of m directed edges, and a positive weight function $\omega: E\to\mathbb{R}$ such that $\omega(e)>0$ for every $e\in E$. We are interested in the *relative* (and not absolute) influence between vertices, so we shall initially fix the weights of self loops to 1, thus defining a "unit of influence", and assume that all other weights are relativized to that unit, and next *normalize* the weight function ω so that $\sum_{e\in E}\omega(e)=|E|=m$. For a set of edges $E'\subseteq E$, define the weight of E' as $\omega(E')=\sum_{e\in E'}\omega(e)$. Given an undirected network, we consider each undirected edge as two equal weight directed edges. Given an unweighted network, we consider all edges to have weight one.

For every vertex v and set of vertices X, let the set of directed edges connecting v to vertices in X be denoted by $E(v,X)$. Similarly, for vertex sets $X,Y\subseteq V$, let $E(X,Y)$ denote the set of directed edges connecting vertices in X to vertices in Y. Based on the edge weights, we define the influence of X on Y, for $X,Y\subseteq V$, as

$$\mathcal{I}(X,Y) \;=\; \omega(E(X,Y))\ . \tag{1}$$

Note that in general $\mathcal{I}(X,Y)\neq\mathcal{I}(Y,X)$. However, if the social network is undirected then $\mathcal{I}(X,Y)=\mathcal{I}(Y,X)$ for every $X,Y\subseteq V$. In addition we define the *total power* of a set X to be

$$\mathcal{I}(X) \;=\; \mathcal{I}(X,X)+\mathcal{I}(X,V\setminus X)\ . \tag{2}$$

Given a core-periphery partition $(\mathcal{C},\mathcal{P})$ of V, the edge set E can be partitioned into four disjoint edge sets $E(\mathcal{C},\mathcal{C}),E(\mathcal{C},\mathcal{P}),E(\mathcal{P},\mathcal{C})$ and $E(\mathcal{P},\mathcal{P})$. Looking at the *adjacency matrix* $A(G)$ of the core-periphery network G [7], these sets correspond to the four basic parts of the *block-model representation* [15] of $A(G)$.

Elite-Periphery Axioms. We now propose three simple axioms that capture what we consider to be basic structural properties required of the core-periphery partition $(\mathcal{E}, \mathcal{P})$ in *elite-centered* social networks[2], namely, *dominance*, *robustness*, and *compactness*. To state our axioms we first define three corresponding quantitative measures for the dominance, robustness, and compactness of the elite $\mathcal{E}$ in a given $(\mathcal{E}, \mathcal{P})$ partition.

1. **Dominance:** The first measure, referred to as the *elite dominance*, concerns the balance of forces exerted on the *periphery*; namely, it compares the influence of the elite on the periphery with the internal influence of the periphery. Formally,

$$\mathrm{dom}(\mathcal{E}) = \mathcal{I}(\mathcal{E}, \mathcal{P}) / \mathcal{I}(\mathcal{P}, \mathcal{P}),$$

and the first axiom is:

(A1) Elite-Dominance: $\mathrm{dom}(\mathcal{E}) \geq c_d$, for a fixed constant $c_d > 0$

This axiom states that the elite *dominates* the rest of society, namely, that the *external* influence maintained by the elite $\mathcal{E}$ on the periphery $\mathcal{P}$ is higher (or at least not significantly lower) than the *internal* influence that the periphery has on itself. Such high dominance is essential for the elite to be able to maintain its superior status in society.

2. **Robustness:** The second measure, referred to as the *elite robustness*, concerns the forces exerted on the *elite*, namely, it compares the internal influence of the elite with the influence of the periphery on the elite. Formally,

$$\mathrm{rob}(\mathcal{E}) = \mathcal{I}(\mathcal{E}, \mathcal{E}) / \mathcal{I}(\mathcal{P}, \mathcal{E}),$$

and the second axiom is:

(A2) Elite-Robustness: $\mathrm{rob}(\mathcal{E}) \geq c_r$, for a fixed constant $c_r > 0$

This axiom claims that elite is *robust*; to maintain its cohesiveness and be able to stick to its opinions, the elite must be able to resist "outside" pressure in the form of the periphery's external influence. To achieve that, the *internal* influence of the elite $\mathcal{E}$ on itself must be greater (or at least not significantly less) than the *external* influence exerted on $\mathcal{E}$ by the periphery.

3. **Compactness:** The third measure, referred to as the *elite compactness*, concerns the *disproportionality* between the elite's *power* and *size*. Let $\delta_X = \frac{\log \mathcal{I}(X)}{\log |X|}$ denote the *log-density* of a set $X \subseteq V$. Then

$$\mathrm{comp}(\mathcal{E}) = \delta_{\mathcal{E}} / \delta_V,$$

and the third axiom is:

(A3) Elite-Compactness: $\mathrm{comp}(\mathcal{E}) \geq 1 + c_c$, for a fixed constant $c_c > 0$

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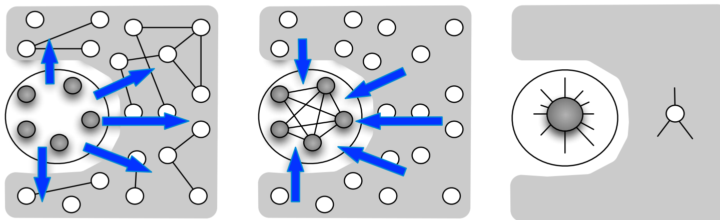
[2] To emphasize our focus on networks whose core is an elite, we denote the core set of the partition by $\mathcal{E}$ rather than $\mathcal{C}$.



(A1) Dominance (A2) Robustness (A3) Compactness

Fig. 2 Graphical illustration of the three axioms. Elite vertices are gray. (A1) The elite's external influence (blue edges), $\mathcal{I}(\mathcal{E}, \mathcal{P})$, dominates the periphery's internal influence (black edges), $\mathcal{I}(\mathcal{P}, \mathcal{P})$. (A2) The internal influence of the elite, $\mathcal{I}(\mathcal{E}, \mathcal{E})$, is robust to the periphery's external influence, $\mathcal{I}(\mathcal{P}, \mathcal{E})$. (A3) The elite is more compact and its average individual is more powerful than an average individual in the society.

This axiom states that the elite members are more compact (or dense) than the entire network. This means that on average an elite member holds significantly more power than an arbitrary member of society.

The three axioms are illustrated graphically in Figure 2. We say that a family of n-vertex networks G_n, for growing n, satisfies the axiom A if there exists some n_0 such that G_n satisfies A for every $n \geq n_0$.

Before showing the implications of these axioms we show that the three axioms are independent. For any two axioms out of the three, there exist a social network and a core-periphery partition that satisfies the two axioms but not the third. More formally, we have the following.

Theorem 1 (Axiom independence). *Axioms* (*A*1)*,* (*A*2)*,* (*A*3) *are independent, namely, assuming any two of them does not imply the third.*

Proof Sketch. We prove the theorem by considering three examples of families of n-vertex (undirected, unweighted) networks and core-periphery partitions for them, described next. Each of these partitions satisfies two of the axioms, but violates the third, implying axiom independence. The first network and partition (Fig. 3(a)), depict a core that is robust and compact but whose dominance tends to zero as the network size n grows to infinity. The second example (Fig. 3(b)) describes a core that is dominant and compact but whose robustness tends to zero as the network grows. The last example (Fig. 3(c)) describes a core that is dominant and robust its compactness dominance tends to one as the network size n grows to infinity. i.e., the average degree of core members and periphery members is almost the same. □

We observe that, as one can easily verify, there are certain networks for which no core-periphery partition satisfies all three axioms (A1), (A2), and (A3) simultaneously. Interestingly, in the special case of undirected networks, axioms (A1) and (A2) are "inversely dependent", namely, every network and

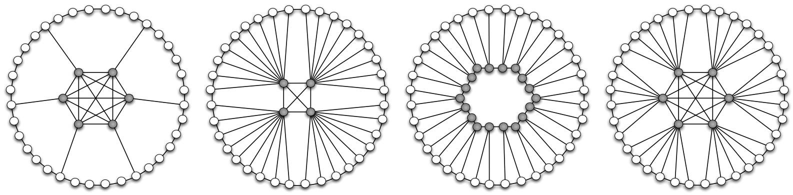


Fig. 3 Network examples demonstrating the independence of the axioms (the gray vertices form the core). (a) The core is robust and compact but not dominant. (b) The core is dominant and compact but not robust. (c) The core is dominant and robust but not compact. (d) An example of a network satisfying all three axioms.

every core-periphery partition must satisfy at least one of them. This implies that there are no unweighted networks that disobey all three axioms.

3 The Size of the Elite

We can now use our axioms to provide bounds on the elite size. The class of *elite-centered social networks* consists of social networks that admit a core-periphery partition satisfying all three axioms. Our main theorm shows that elite-centered social networks have a sublinear elite. Formally,

Theorem 2 (Elite Size). *If* $(\mathcal{E}, \mathcal{P})$ *satisfies the dominance, robustness, and compactness axioms (A1), (A2), and (A3), then the elite size is* sublinear *in the size of society, namely,*

$$c \cdot n^{\frac{\delta_V}{\delta_\mathcal{E}}} \le |\mathcal{E}| \le n^{\frac{1}{1+c_c}} \ .$$

We find it remarkable that three simple and intuitive assumptions lead to such a strong implication on the elite size. Note that Theorem 2 is controversial to the common belief that the elite size converges to a linear fraction of a society's size (most recently claimed to be 1% [23]). This discrepancy may perhaps be attributed to the fact that our axiom-based approach characterizes the elite differently than in previous approaches.

To prove Theorem 2 we first observe that every network satisfying the axioms has the following properties.

Lemma 1. *If* $(\mathcal{E}, \mathcal{P})$ *satisfies the dominance and robustness axioms (A1) and (A2), then the total influence of the elite is at least a fraction of the total influence in society, namely, for some constant* $c_1 > 0$,

$$c_1 \cdot m \ \le\ \mathcal{I}(\mathcal{E}) \ \le\ m.$$

Proof. We first note that since $\omega(E) = m$ we have:

$$\mathcal{I}(\mathcal{E},\mathcal{E}) + \mathcal{I}(\mathcal{E},\mathcal{P}) + \mathcal{I}(\mathcal{P},\mathcal{E}) + \mathcal{I}(\mathcal{P},\mathcal{P}) \ = \ m, \qquad (3)$$

so $\mathcal{I}(\mathcal{E}) \le m$. By Eq. (3) and Axioms (A1) and (A2),

$$\mathcal{I}(\mathcal{E},\mathcal{E}) + \mathcal{I}(\mathcal{E},\mathcal{P}) + \frac{\mathcal{I}(\mathcal{E},\mathcal{E})}{c_r} + \frac{\mathcal{I}(\mathcal{E},\mathcal{P})}{c_d} \;\ge\; m,$$

so
$$\left(1 + \frac{1}{\min(c_r, c_d)}\right)\mathcal{I}(\mathcal{E}) \;\ge\; m,$$

hence $\mathcal{I}(\mathcal{E}) \ge c_1 \cdot m$ for constant $c_1 = (1 + \frac{1}{\min(c_r,c_d)})^{-1}$. □

Using Axiom (A3), the elite size can now be tightly bounded in terms of the compactness parameters $\delta_{\mathcal{E}}$ and δ_V, establishing Theorem 2.

Proof (Of Theorem 2). Recalling that $m = n^{\delta_V}$ and $\mathcal{I}(\mathcal{E}) = |\mathcal{E}|^{\delta_{\mathcal{E}}}$, the proof follows directly from Lemma 1 and Axiom (A3), which states that $\delta_V/\delta_{\mathcal{E}} \le 1/(1 + c_c)$. □

In reality, the question of an upper bound for the "typical" elite is unanswered: does the "universal" size of elites (if it exists) converge to a linear, or a sublinear, function of the network size? In [4] we present evidence that many social networks are elite-centered (namely, satisfy our axioms), which indicates sublinear elites.

Interestingly, if Axiom (A3) does not hold, then it is possible for the elite to be of linear size. This will be the case, for example, in social networks where Dunbar's theory holds [13]. Dunbar suggested a cognitive limit to the number of people with whom one can maintain stable social relationships. If this is the case, then one can show that an elite that satisfies Axioms (A1) and (A2) *must* be of linear size. One can even claim a slightly stronger result, stating that the elite's average degree is bounded from above by a constant times the average degree in the network (which necessitates linear elite size). Formally, we state the following.

Lemma 2. *If $(\mathcal{E},\mathcal{P})$ satisfies the dominance and robustness axioms (A1) and (A2), but has, for some constant c,*

$$\mathcal{I}(\mathcal{E})/|\mathcal{E}| \le c \cdot \mathcal{I}(V)/|V|,$$

then $|\mathcal{E}| \ge c_3 \cdot n$ for some constant c_3.

Proof. By Lemma 1, and since here $\mathcal{I}(\mathcal{E}) \le |\mathcal{E}|cm/n$, we have $|\mathcal{E}| \ge c_1 n/c$. □

We now turn lower bounds on the elite size. How small can the elite be while still maintaining its power and satisfying the axioms? Let us first observe that in the general case of weighted or directed networks (such as twitter for example), no nontrivial lower bounds hold, and the network may have an extremely small elite (possibly even constant size) that satisfies our axioms.

Lemma 3. *There are (directed or weighted) networks for which an $(\mathcal{E}, \mathcal{P})$ partition satisfies the dominance, robustness, and compactness axioms (A1),(A2), and (A3), and the elite has a constant number of members.*

Proof. For directed networks, a classical example is the *star* graph, where a single vertex (the center, forming the elite) has a directed edge to each of the periphery vertices (with no incoming edges). Clearly the star center dominates the periphery, and it is robust and compact.

Next consider undirected weighted graphs. Consider a tree network with $n = 2k$ vertices, so $m = 4k - 1$ (including self-loops). The weight of each self-loop is 1, totaling $2k$. Now the tree is constructed from two stars, with uniform edge weights of 1/2, plus an edge of weight k connecting the two centers of the stars. It is easy to check that the sum of the edge weights is m. The elite consisting of the two star centers satisfies all three axioms. □

In contrast, in undirected unweighted networks, the following lower bound can be established for elite size.

Theorem 3. *In an unweighted and undirected network G with a core-periphery partition $(\mathcal{E}, \mathcal{P})$, if the core $\mathcal{E}$ satisfies the dominance and robustness axioms (A1) and (A2), then its size satisfies $|\mathcal{E}| \geq c_4 \cdot \sqrt{m}$ for some constant $c_4 > 0$.*

Proof. In the undirected case $\mathcal{I}(\mathcal{E}, \mathcal{P}) = \mathcal{I}(\mathcal{P}, \mathcal{E})$, so

$$\mathcal{I}(\mathcal{E}, \mathcal{E}) + \mathcal{I}(\mathcal{E}, \mathcal{P}) + \mathcal{I}(\mathcal{P}, \mathcal{P}) \;=\; m. \tag{4}$$

By the two axioms and since G is undirected, we have

$$\mathcal{I}(\mathcal{E}, \mathcal{E}) \;\geq\; c_r \cdot \mathcal{I}(\mathcal{P}, \mathcal{E}) \;\geq\; c_r c_d \cdot \mathcal{I}(\mathcal{P}, \mathcal{P}).$$

Combining this with Eq. (4), we obtain

$$m \;\leq\; \left(1 + \frac{1}{c_r} + \frac{1}{c_r c_d}\right) \mathcal{I}(\mathcal{E}, \mathcal{E}).$$

Hence, when setting $c_2 = (1 + 1/c_r + 1/(c_r c_D))^{-1}$ it holds that

$$\mathcal{I}(\mathcal{E}, \mathcal{E}) \;\geq\; c_2 \cdot m. \tag{5}$$

Graph-theoretical considerations dictate that $\mathcal{I}(\mathcal{E}, \mathcal{E}) \leq \binom{|\mathcal{E}|}{2} \leq |\mathcal{E}|^2$, implying that $|\mathcal{E}| \geq \sqrt{\mathcal{I}(\mathcal{E}, \mathcal{E})}$. Combined with Eq. (5), the theorem follows. □

One can also show an example of what we call a *purely elitistic society*, where the elite reaches its minimum possible size of $\Theta(\sqrt{m})$ in undirected, unweighted networks. See Figure 3(d) for an illustrative example.

We remark that in addition to the theoretical results on elite axioms and properties, we also studied some real networks, in order to examine the extent to which our axioms manifest in reality, and provide evidence for the existence of real elite-centered social networks [4].

4 Related Work

The axiomatic approach has been used successfully in many fields of science, such as mathematics, physics, economy, sociology and computer science. See [2, 17] for two examples in areas related to ours.

A variety of notions for measuring influence in a network and for core-periphery partitions have been developed in the past (see the recent survey [9]). Borgatti and Everett [5] measure the similarity between the adjacency matrix of a graph and the block matrix $\binom{1\ 1}{1\ 0}$. This captures the intuition that social networks have a dense, cohesive core and a sparse, disconnected periphery. an intuition also reflected in the axioms postulated herein.

Methods for identifying core-periphery structures and partitioning networks include algorithms for detecting (along with statistical tests for verifying) a-priori hypotheses [6], a coefficient measuring if a network exhibits a core-periphery dichotomy [19], a method for extracting cores based on a modularity parameter [11], a centrality measure computed as a continuous value along a core-periphery spectrum [24], a coreness value attributed to each vertex, qualifying its position and role based on random walks [12], a detection method using spectral analysis and geodesic paths [10], and a decomposition method using statistical inference [26]. The recent [25] argues that the core-periphery structure is simply the result of several overlapping communities and proposes a community detection method coping with overlap. None of these works consider the asymptotic size of a core/elite and the possibility that its size is sublinear in the population size.

One of the first papers to focus on the fact that the highest degree vertices are well-connected [27] coined the term *rich-club coefficient* for the density of the vertices of degree k or more. Mislove et al. [20] defined the *core* of a network to be any (minimal) set of vertices that satisfies two properties. First, the core must be essential for ensuring the connectivity of the network (i.e., removing it breaks the remaining vertices into many small, disconnected clusters). Second, the core must be strongly connected with a relatively small diameter. Mislove et al. used an approximation technique based on removing increasing numbers of the highest degree vertices (rich clubs) and analyzing the connectivity of the remaining graph. The graphs studied in [20] have a densely connected core comprising of between 1% and 10% of the highest degree vertices, such that removing this core completely disconnects the graph. Thus, the authors provide further evidence that rich clubs are crucial in social networks and satisfy their core properties.

A very different perspective is offered in [18]; a network formation game is studied, where benefits from connections exhibit decreasing returns and decay with network distance. In line with our axioms, the equilibria of this game form core-periphery structures. Another network formation game is developed in [16], where players invest in information acquisition. The authors show what they call "The Law of the Few": the economic forces are leading to a robust equilibrium where the majority of individuals to obtain most of the information from a very small subset of the group. The size of this subset is sublinear, so its fraction out of the population converges to zero. While these results hold under a more specific set of assumptions, they confirm the results derived from our more general axioms.

Recently, [3] used ideas presented in this paper to study the influence properties of the set of *founders*, the vertices arriving first, in the preferential attachment model of [8] under different model parameters. If the number of edges in the model is linear in the number of vertices (i.e., edge and vertex events happen with constant probability), then networks generated by preferential attachment must have a *linear* size founders set to be dominant, implying that this set will not satisfy our third axiom. On the other hand, if the number of edges in the model is super-linear in the number of vertices (i.e., the probability of vertex events decreases to zero over time), then the generated networks feature a *sublinear* size founders set that is dominant. This also demonstrates that both linear and sublinear cores are possible, depending on the network type.

5 Conclusion

In this article, we provide axioms modeling the influence relationships between the elite and the periphery. We prove that for a core-periphery partition that satisfies our axioms, the core forms an elite of sublinear size in the number of network vertices. In particular, this means that an elite is much smaller than a constant fraction of the network, evidence of which is often observed in the widening gap between the very rich and the rest of society.

Some of the above findings may have been known on an anecdotal level, or may seem obvious; our axioms allow us to quantify the forces at play and compare different core-periphery partitions. For example, it is shown in [3] that also in the well-accepted preferential attachment model, that founder cores might not satisfy all our axioms. Thus, it is of a major interest to find evolutionary models in which elites as described here emerge naturally.

Our results not only advance the theoretical understanding of the elite of social structures, but may also help to improve infrastructure and algorithms targeted at online social networks, e.g., to organize institutions better, or identify sources of power in social networks in general.

References

1. Alberich, R., Miro-Julia, J., and Rosselló, F. Marvel universe looks almost like a real social network. *arXiv preprint cond-mat/0202174* (2002).
2. Andersen, R., Borgs, C., Chayes, J. T., Feige, U., Flaxman, A. D., Kalai, A., Mirrokni, V. S., and Tennenholtz, M. Trust-based recommendation systems: an axiomatic approach. In *Proc. WWW* (2008), pp. 199–208.

3. Avin, C., Lotker, Z., Nahum, Y., and Peleg, D. Core size and densification in preferential attachment networks. In *Proc. 42nd Int. Colloq. on Automata, Languages, and Programming (ICALP), 2015* (2015), pp. 492–503.
4. Avin, C., Lotker, Z., Peleg, D., Pignolet, Y. A., and Turkel, I. Elites in social networks: An axiomatic approach. `http://bit.ly/2fqLPUT`, 2016.
5. Borgatti, S., and Everett, M. Models of core/periphery structures. *Social networks 21*, 4 (2000), 375–395.
6. Borgatti, S., Everett, M., and Freeman, L. Ucinet: Software for social network analysis. *Harvard Analytic Technologies 2006* (2002).
7. Borgatti, S., Everett, M., and Johnson, J. *Analyzing Social Networks.* London: Sage Publications, 2013.
8. Chung, F. R. K., and Lu, L. *Complex graphs and networks.* AMS, 2006.
9. Csermely, P., London, A., Wu, L.-Y., and Uzzi, B. Structure and dynamics of core/periphery networks. *J. Complex Networks 1*, 2 (2013), 93–123.
10. Cucuringu, M., Rombach, P., Lee, S. H., and Porter, M. A. Detection of core-periphery structure in networks using spectral methods and geodesic paths. *European J. Applied Mathematics* (2016), 1–42.
11. Da Silva, M. R., Ma, H., and Zeng, A.-P. Centrality, network capacity, and modularity as parameters to analyze the core-periphery structure in metabolic networks. *Proc. IEEE 96*, 8 (2008), 1411–1420.
12. Della Rossa, F., Dercole, F., and Piccardi, C. Profiling core-periphery network structure by random walkers. *Scientific reports 3* (2013).
13. Dunbar, R. Neocortex size as a constraint on group size in primates. *J. Human Evolution 22*, 6 (1992), 469 – 493.
14. Facundo, A., Atkinson, A. B., Piketty, T., and Saez, E. The world top incomes database, 2013.
15. Faust, K., and Wasserman, S. Blockmodels: Interpretation and evaluation. *Social Networks 14* (1992), 5–61.
16. Galeotti, A., and Goyal, S. The law of the few. *The American Economic Review* (2010), 1468–1492.
17. Geiger, D., Paz, A., and Pearl, J. Axioms and algorithms for inferences involving probabilistic independence. *Inf. & Computat. 91*, 1 (1991), 128 – 141.
18. Hojman, D. A., and Szeidl, A. Core and periphery in networks. *J. Economic Theory 139*, 1 (2008), 295–309.
19. Holme, P. Core-periphery organization of complex networks. *Physical Review E 72*, 4 (2005), 046111.
20. Mislove, A., Marcon, M., Gummadi, K. P., Druschel, P., and Bhattacharjee, B. Measurement and analysis of online social networks. In *Proc. 7th ACM SIGCOMM Conf. on Internet measurement* (2007), ACM, pp. 29–42.
21. Oxfam International. Working for the few: Political capture and economic inequality, January 2014.
22. Pareto, V. *The mind and society.* AMS, 1935.
23. Piketty, T. *Capital in the Twenty-first Century.* Harvard Univ. Press, 2014.
24. Rombach, M. P., Porter, M. A., Fowler, J. H., and Mucha, P. J. Core-periphery structure in networks. *SIAM J. Applied mathematics 74*, 1 (2014), 167–190.
25. Yang, J., and Leskovec, J. Overlapping communities explain core–periphery organization of networks. *Proc. IEEE 102*, 12 (2014), 1892–1902.
26. Zhang, X., Martin, T., and Newman, M. E. J. Identification of core-periphery structure in networks. *Phys. Rev. E 91* (Mar 2015), 032803.
27. Zhou, S., and Mondragón, R. The rich-club phenomenon in the internet topology. *IEEE Communications Lett. 8* (2004), 180–182.